Homework #3

November 27, 2009

The assignments are due Friday, December 4 at 11AM in class.

Exercise 1 (3pt) Prove the Lindenbaum’s Lemma: If $\Sigma$ is a $K$-consistent set of formulas then there is a $K$-maximal consistent set of formulas $\Sigma^+$ such that $\Sigma \subseteq \Sigma^+$.

Proof Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be an enumeration of the formulas of our language. We define the set $\Sigma^+$ as the union of a chain of $K$-consistent sets as follows:

$$\Sigma_0 = \Sigma$$

$$\Sigma_{n+1} = \begin{cases} 
\Sigma_n \cup \{\varphi_n\} & \text{if this is } K\text{-consistent} \\
\Sigma_n \cup \{\neg \varphi_n\} & \text{otherwise}
\end{cases}$$

$$\Sigma^+ = \bigcup_{n \geq 0} \Sigma_n$$

To finish the proof you need to show the following properties:

1. $\Sigma \subseteq \Sigma^+$
2. $\Sigma^+$ is $K$-consistent
3. $\Sigma^+$ is maximal

Exercise 2 (5pt) Prove the valuation lemma: For any maximal $K$-consistent set of formulas $\Gamma$ (that is, for any node in the canonical model), for any formula $\varphi$:

$$M_K, \Gamma \models \varphi \iff \varphi \in \Gamma.$$  

Hint: induction on $\varphi$.

Exercise 3 (2pt) Show that the following rule is admissible in $K$: if $\vdash K \Box \alpha \lor \Box \beta$, then $\vdash K \alpha$ or $\vdash K \beta$. (Hint: $K$ is complete)