COMPLEXITY OF QUANTIFIERS

Jakub Szymanik

ILLC UvA

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OUTLINE

1. Motivations
2. Simple Quantifiers
3. Complex Quantifiers
4. Summary

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1. Motivations
2. Simple Quantifiers
   - Monadic Generalized Quantifiers
   - Automata for Monadic Quantifiers
   - Experimental Data
     - Quantifiers and Counting
     - Comparing Quantifiers
3. Complex Quantifiers
   - Standard Quantifier Combinations
   - Branching Quantifiers
   - Quantified Reciprocals
4. Conclusions

COMPUTABILITY AND COGNITION

- A cognitive task is a computational task.
- Marr's levels: computational, algorithmic, neurological.
- Today computational restrictions are taken seriously.
    - Tsotsos, Analyzing vision at the complexity level, 1990.
    - Frixione, Tractable competence, 2001.
    - van Rooij, The tractable cognition thesis, 2008.

MEANING AS ALGORITHM

- Ability of understanding sentences.
- Capacity of recognizing their truth-values.
- Fregean tradition.
- Meaning is a procedure for finding extension in a model.
- Adopted often with psychological motivations:
  - Suppes, Variable-free semantics with remark on procedural extensions, 1982.
  - Lambalgen & Hamm, The Proper Treatment of Events, 2005.

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Simple quantifier sentences

- Every poet has low self-esteem.
- Some dean danced nude on the table.
- At least 3 grad students prepared presentations.
- An even number of the students saw a ghost.
- Most of the students think they are smart.
- Less than half of the students received good marks.

Lindström definition

Definition

A monadic generalized quantifier of type (1,1) is a class Q of structures of the form $M = (U, A_1, A_2)$, where $A_1, A_2 \subseteq U$. Additionally, Q is closed under isomorphism.

A FEW EXAMPLES

- some $= \{(U, A, B) : A, B \subseteq U \wedge A \cap B \neq \emptyset\}$
- all $= \{(U, A, B) : A, B \subseteq U \wedge A \subseteq B\}$
- exactly m $= \{(U, A, B) : A, B \subseteq U \wedge \text{card}(A \cap B) = m\}$
- even $= \{(U, A, B) : A, B \subseteq U \wedge \text{card}(A \cap B) = k \times 2\}$
- most $= \{(U, A, B) : \text{card}(A \cap B) > \text{card}(A - B)\}$

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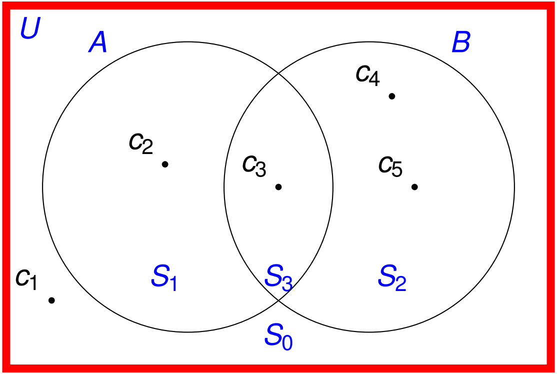
HOW DO WE ENCODE MODELS?

- Restriction to finite models of the form $M = (U, A, B)$.
- List of all elements of the model: $c_1, \ldots, c_5$.
- Labeling every element with one of the letters: $a_{\bar{A}\bar{B}}$, $a_{A\bar{B}}$, $a_{\bar{A}B}$, a_{AB}, according to constituents it belongs to.
- Result: the word $\alpha_M = a_{\bar{A}\bar{B}}a_{A\bar{B}}a_{AB}a_{\bar{A}B}a_{\bar{A}B}$.
- α_M describes the model in which: $c_1 \in \bar{A}\bar{B}, c_2 \in A\bar{B} c_3 \in AB, c_4 \in \bar{A}B, c_5 \in \bar{A}B$.
- The class Q is represented by the set of words describing all elements of the class.

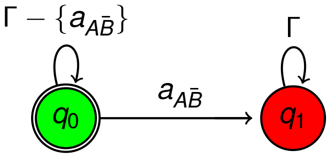
Illustration



This model is uniquely described by $\alpha_M = a_{\bar{A}\bar{B}} a_{A\bar{B}} a_{AB} a_{\bar{A}B} a_{\bar{A}B}$

ARISTOTELIAN QUANTIFIERS

"all", "some", "no", and "not all"

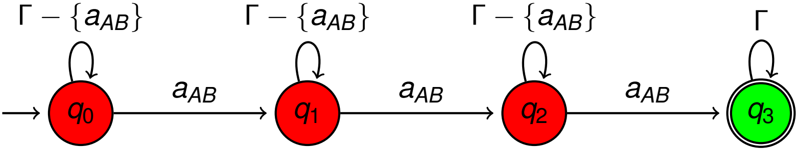


Finite automaton recognizing L_{All}

$$L_{\mathrm{All}} = \{\alpha \in \Gamma^* : \#a_{A\bar{B}}(\alpha) = 0\}$$

Cardinal quantifiers

E.g. “at least 3”, “at most 7”, and “between 8 and 11”

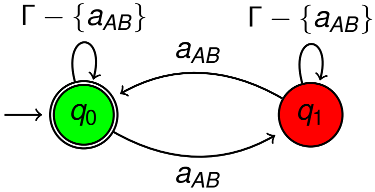


Finite automaton recognizing $L_{\text{At least three}}$

$$L_{\text{At least three}} = \{\alpha \in \Gamma^* : \#a_{AB}(\alpha) \geq 3\}$$

PARITY QUANTIFIERS

E.g. “an even number”, “an odd number”



Finite automaton recognizing L_{Even}

$$L_{\text{Even}} = \{\alpha \in \Gamma^* : \#a_{AB}(\alpha) \text{ is even}\}$$

PROPORTIONAL QUANTIFIERS

- E.g. “most”, “less than half”.
- Most *As are B* iff card$(A \cap B) >$ card$(A - B)$.
- $L_{\text{Most}} = \{\alpha \in \Gamma^* : \#a_{AB}(\alpha) > \#a_{A\bar{B}}(\alpha)\}$.
- There is no finite automaton recognizing this language.
- We need internal memory.
- A push-down automata will do.

What does it mean that class of monadic quantifiers is recognized by class of devices?

Definition

Let $\mathcal{D}$ be a class of recognizing devices,
Ω a class of monadic quantifiers.
We say that $\mathcal{D}$ accepts Ω if and only if
for every monadic quantifier Q:

$$\mathsf{Q} \in \Omega \iff \text{there is device } A \in \mathcal{D}(A \textit{ accepts } L_{\mathsf{Q}}).$$

In General

Definability	Examples	Recognized by
FO	"all" "at least 3"	acyclic FA
FO(D_n)	"an even number"	FA
PrA	"most", "less than half"	PDA

Quantifiers, definability, and complexity of automata

van Benthem, Essays in logical semantics, 1986.

Mostowski, Computational semantics for monadic quantifiers, 1998.

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PREVIOUS INVESTIGATIONS

- Quantifiers widely treated in cognitive psychology.
- Brain activity during the comprehension of:
  - FO-quantifiers vs. higher-order quantifiers;
  - All quantifies are associated with numerosity;
  - Higher-order activate working-memory capacity;
  - Computational differences missed;
  - "Even" is higher-order but FA-computable.

- McMillan et al., "Neural basis for generalized quantifiers comprehension", 2005.
- Szymanik, "A Note on a Neuroimaging Study of Natural Language Quantifiers Comprehension", 2007.

General idea

- Compare RT wrt the following classes of quantifiers:
  - recognized by acyclic FA (first-order);
  - not first-order recognized by FA (parity);
  - recognized by PDA but not FA.
- Additionally:
  - Aristotelian vs. cardinal quantifiers of higher rank.

PREDICTIONS

- RT will increase along with the computational resources.
- Aristotelian qua. < parity qua. < proportional qua.
- Aristotelian qua. < cardinal qua. of high rank.
- Parity qua. < cardinal qua. of high rank.

PARTICIPANTS

- 40 native Polish-speaking adults (21 female).
- Volunteers: undergraduates from the University of Warsaw.
- The mean age: 21.42 years (SD = 3.22).
- Each participant tested individually.

MATERIALS

80 grammatically simple propositions in Polish, like:

1. Some cars are red.
2. More than 7 cars blue.
3. An even number of cars is yellow.
4. Less than half of the cars are black.

Materials continued

Most of the cars are yellow.

An example of a stimulus used in the first study

PROCEDURE

- 8 different quantifiers divided into four groups.
  - “all” and “some”;
  - “odd” and “even”;
  - “less than 8” and “more than 7”;
  - “less than half” and “more than half”.
- Each quantifier was presented in 10 trials.
- The sentence true in the picture in half of the trials.
- Quantity of target items near the criterion of validation.
- Practice session followed by the experimental session.
- Each quantifier problem was given one 15.5 s event.
- Subjects were asked to decide the truth-value.

ANALYSIS OF ACCURACY

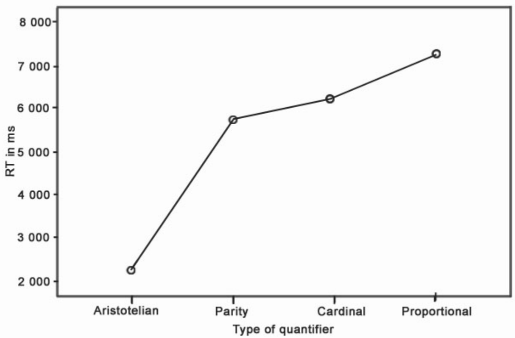
Quantifier group	Examples	Percent
Aristotelian FO	all, some	99
Parity	odd, even	91
Cardinal FO	less than 8, more than 7	92
Proportional	less than half, more than half	85

The percentage of correct answers

TO SUM UP

- Increase in RT was determined by the quantifier type ($F(2.4, 94.3) = 341.24; p < 0,001; \eta^2 = 0.90$)
- Pairwise comparisons: all four types of quantifiers differed significantly from one another.
- The mean reaction time increased as follows: Aristotelian, parity, cardinal, proportional.

Comparison of reaction times



Average reaction times in each type of quantifiers

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Polyadic GQs

Definition

A *generalized quantifier* Q of type $t = (n_1, \ldots, n_k)$ is a functor assigning to every set M a k-ary relation Q_M between relations on M such that if $(R_1, \ldots, R_k) \in Q_M$ then R_i is an n_i-ary relation on M, for $i = 1, \ldots, k$. Additionally, Q is preserved by bijections.

Definition

If for all i the relation R_i is unary, i.e. it denotes a subset of the universe, then we say that the quantifier is *monadic*. Otherwise, it is *polyadic*.

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Boolean Combinations

1. At least 5 or at most 10 departments can win EU grants.
2. Between 100 and 200 students started in the marathon.
3. Not all students passed.
4. All students did not pass.

Boolean Combinations

Definition

Let Q, Q′ be generalized quantifiers, both of type $(n_1, \ldots, n_k)$. We define:

$$(Q \wedge Q')_M[R_1, \ldots, R_k] \iff Q_M[R_1, \ldots, R_k] \text{ and } Q'_M[R_1, \ldots, R_k]$$

$$(Q \vee Q')_M[R_1, \ldots, R_k] \iff Q_M[R_1, \ldots, R_k] \text{ or } Q'_M[R_1, \ldots, R_k]$$

$$(\neg Q)_M[R_1, \ldots, R_k] \iff \text{not } Q_M[R_1, \ldots, R_k]$$

$$(Q\neg)_M[R_1, \ldots, R_k] \iff Q_M[R_1, \ldots, R_{k-1}, M - R_k]$$

Iteration

1. Most logicians criticized some papers.
2. $\mathrm{It}(\mathrm{most}, \mathrm{some})$[Logicians, Papers, Criticized].

Definition

$\mathrm{It}(\mathrm{Q}, \mathrm{Q}')[A, B, R] \iff \mathrm{Q}[A, \{a \mid \mathrm{Q}'(B, R_{(a)})\}]$, where $R_{(a)} = \{b \mid R(a, b)\}$.

Cumulation

1. Eighty professors taught sixty courses at ESSLLI’08.

Definition

$$\mathrm{Cum}(\mathsf{Q}, \mathsf{Q}')[A, B, R] \iff$$
$$\mathrm{It}(\mathsf{Q}, \mathrm{some})[A, B, R] \wedge \mathrm{It}(\mathsf{Q}', \mathrm{some})[B, A, R^{-1}]$$

Basic Operations are Tractable

Theorem

Let Q *and* Q' *be generalized quantifiers computable in PTIME with respect to the size of a universe. Then the quantifiers: (1)* $\neg\mathsf{Q}$*; (2)* $\mathsf{Q}\neg$*; (3)* $\mathsf{Q} \wedge \mathsf{Q}'$*; (4)* $\mathrm{It}(\mathsf{Q}, \mathsf{Q}')$*; (5)* $\mathrm{Cum}(\mathsf{Q}, \mathsf{Q}')$*; (6)* $\mathrm{Res}(\mathsf{Q})$ *are PTIME computable.*

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Possibly branching sentences

1. Some relative of each villager and some relative of each townsman hate each other.
2. Some book by every author is referred to in some essay by every critic.
3. Most villagers and most townsmen hate each other.
4. One third of villagers and half of townsmen hate each other.
5. 5 villagers and 7 townsmen hate each other.

ITERATION AND BRANCHING

Most girls and most boys hate each other.

LINEAR: $\text{most } x\ [G(x), \text{most } y\ (B(y), H(x, y))].$

BRANCHING: $\begin{array}{l} \text{most } x : G(x) \\ \text{most } y : B(y) \end{array} H(x, y).$

$\exists A \exists A' [\text{most}(G, A) \wedge \text{most}(B, A') \wedge \forall x \in A \forall y \in A' H(x, y)].$

Iteration and Branching

Most girls and most boys hate each other.

Linear: most x $[G(x), \text{most } y\ (B(y), H(x,y))]$.

Branching: $\begin{matrix} \text{most } x : G(x) \\ \text{most } y : B(y) \end{matrix}\ H(x,y)$.

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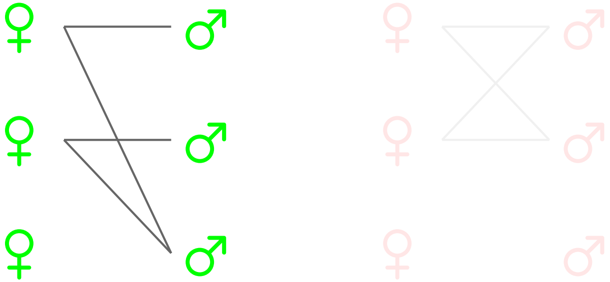
What is the difference?



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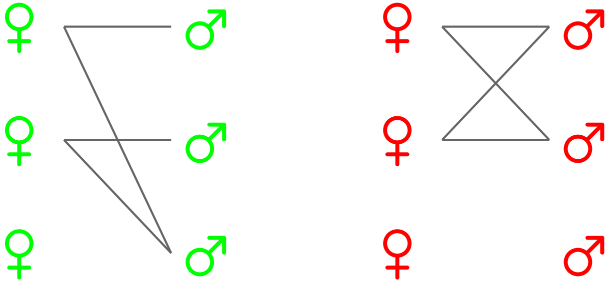

Branching Readings are Intractable

Theorem

Branching sentences are NP-complete.

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Reciprocals are common in English

1. Andi, Jarmo and Jakub laughed at one another.
2. 15 men are hitting one another.
3. Even number of the PMs refer to each other.
4. Most Boston pitchers sat alongside each other.
5. Some pirates were staring at each other in surprise.

Various interpretations

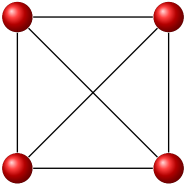
Dalrymple et al. 1998 classifies possible readings.
They explain variations in the meaning by:

Strong Meaning Hypothesis

Reading associated with the reciprocal in a given sentence is the strongest available reading which is consistent with relevant information supplied by the context.

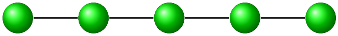
STRONG READING

Even number of the PMs refer to each other.



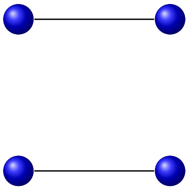
INTERMEDIATE READING

Most Boston pitchers sat alongside each other.



WEAK READING

Some pirates were staring at each other in surprise.



STRONG RECIPROCAL LIFT

Let Q be a monadic monotone increasing quantifier.

DEFINITION

$\mathrm{Ram}_S(\mathrm{Q})(A, R) \iff$

$$\exists X \subseteq A[\mathrm{Q}(X) \wedge \forall x, y \in X(x \neq y \Rightarrow R(x, y))].$$

EXAMPLE

- Even number of the PMs refer to each other indirectly.
- $\mathrm{Ram}_S(\textsc{Even})[\mathrm{MP}, \mathrm{Refer}]$.

INTERMEDIATE RECIPROCAL LIFT

DEFINITION

$$\mathrm{Ram}_{\mathrm{I}}(\mathrm{Q})(A, R) \iff \exists X \subseteq A[\mathrm{Q}(X) \wedge \forall x, y \in X$$
$$(x \neq y \Rightarrow \exists \text{ sequence } z_1, \ldots, z_\ell \in X \text{ such that}$$
$$(z_1 = x \wedge R(z_1, z_2) \wedge \ldots \wedge R(z_{\ell-1}, z_\ell) \wedge z_\ell = y)].$$

EXAMPLE

- Most Boston pitchers sat alongside each other.
- $\mathrm{Ram}_{\mathrm{I}}(\mathrm{most})[\mathrm{Pitcher}, \mathrm{Sit}]$.

Weak reciprocal lift

Definition

$\mathrm{Ram_W}(\mathrm{Q})(A, R) \iff$

$$\exists X \subseteq A[\mathrm{Q}(X) \wedge \forall x \in X \exists y \in X(x \neq y \wedge R(x, y))].$$

Example

- Some pirates were staring at each other in surprise.
- $\mathrm{Ram_W}(\mathrm{some})[\mathrm{Pirate},\ \mathrm{Staring}]$.

Complexity Dichotomy

Theorem

Model-checking for strong reciprocal sentences with proportional quantifiers is NP-complete.

Theorem

If Q is PTIME quantifier, then also $\mathrm{Ram}_{\mathrm{I}}(\mathrm{Q})$ and $\mathrm{Ram}_{\mathrm{W}}(\mathrm{Q})$ are in PTIME.

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General Question

To what extent differences in computational complexity of quantifiers influence their comprehension?

Thank you!