Computational complexity of some Ramsey quantifiers in finite models

Marcin Mostowski
Jakub Szymanik

Institute of Philosophy, Warsaw University
Institute for Logic, Language and Computation, University of Amsterdam

Logic Colloquium 2006
Introduction
Ramsey quantifiers and their complexity

Outline

1 Introduction
- Research motivation
- Ramsey quantifiers
- Computational complexity of quantifiers in finite models
- INDEPENDENT SET and q–BIG CLIQUE

2 Ramsey quantifiers and their complexity

Marcin Mostowski, Jakub Szymanik
Computational complexity of Ramsey quantifiers
Outline

1 Introduction
   - Research motivation
   - Ramsey quantifiers
   - Computational complexity of quantifiers in finite models
   - INDEPENDENT SET and q–BIG CLIQUE

2 Ramsey Quantifiers and Their Complexity
Complexity and linguistic competence

1. Deciding whether some natural language sentence is true or not in a given finite situation.
2. Evaluating complexity of semantic construction is important for better understanding our linguistic competence.
3. Some natural language constructions are $NP$–complete - all known examples explore the idea of BQ.
Some relative of each villager and some relative of each townsman hate each other.

Some book by every author is referred to in some essay by every critic.

Most boys and most girls dated each other.

All of the proofs of NP-completeness for BQ are based on some Ramsey property.
THEOREM

When coloring sufficiently large complete finite graph, one will find a big homogeneous subset, i.e. complete subgraph with all edges of the same colour of arbitrary large finite cardinality.

DEFINITION

A Ramsey quantifier R is a generalized quantifier of the type (2), such that $M \models R_{xy} \varphi(x, y)$ exactly when there is $A \subseteq |M|$ (big relatively to the size of $M$) such that for each $a, b \in A$ $M \models \varphi(a, b)$. 
Quantifiers and complexity

**Definition**

Let $Q$ be of the type $(n)$. By complexity of $Q$ we mean the computational complexity of the class $K_Q$ of finite models such that $M \in K_Q$ if and only if $M \models Qx_1, \ldots, x_n H(x_1, \ldots, x_n)$.

**Definition**

We say that $Q$ is $NP$–hard if $K_Q$ is $NP$–hard. $Q$ is mighty if $K_Q$ is $NP$ and $K_Q$ is $NP$–hard.
Let us consider models of the form $M = (U, E)$, where $E$ is an equivalence relation.

**Definition**

$M \models R_e x y \varphi(x, y)$ means that there is a set $A \subseteq U$ such that $\forall a \in |M| \ \exists b \in A \ E(a, b)$ and for each $a, b \in A \ M \models \varphi(a, b)$.

**Theorem (Mostowski, Wojtyniak 2004)**

$R_e$ is mighty.
**MIGHTY QUANTIFIERS - SECOND EXAMPLE**

Let us consider models of the form $M = (U, V, T)$, where $V, T$ are subsets of $U$.

**Definition**

$M \models \text{BMost } xy \varphi(x, y)$ means that there are sets $A \subseteq U$ and $B \subseteq U$ such that $\text{MOST}x (V(x), A(x)) \land \text{MOST}y (T(y), B(y)) \land \forall x \forall y (A(x) \land B(y) \Rightarrow H(x, y))$.

**Theorem (Sevenster 2006)**

$\text{BMost}$ is mighty.
**Independent Set**

**Definition**

Let $G = (V, E)$ be a graph and $I \subseteq V$. We say that $I$ is independent if there is no $(i, j) \in E$ for any two $i, j \in I$.

**Figure**: Independent sets
**INDEPENDENT SET PROBLEM**

Given graph $G = (V, E)$ and natural number $k$ we must determine whether there is independent set in $G$ of cardinality exactly $k$.

**Theorem**

*INDEPENDENT SET* is NP–complete.
**Definition**

We say that \( A \subseteq V \) is a clique for graph \( G = (V, E) \) if \( A^2 \subseteq E \). Problem: whether in \( G \) exists clique of cardinality \( k \).

**Theorem**

*CLIQUE* is NP–complete.
**q–BIG CLIQUE PROBLEM**

**Definition**

We say that $A \subseteq V$ is $q$–big clique in $G = (V, E)$, if $A$ is clique in $G$ and $\frac{\text{card}(A)}{\text{card}(V)} \geq q$.

**Definition**

Let $G = (V, E)$ and $q \in ]0, 1[ \cap \mathbb{Q}$. $q$–BIG CLIQUE problem is to decide if there is $q$–big clique $A \subseteq V$ in $G$. 
**Theorem**

For $q = \frac{1}{3}$, $q$–BIG CLIQUE is NP–complete.

We can consider graphs divided not only into disjoint triangles, but also complete quadrangles, pentagons, hexagons and so on . . .

**Theorem**

$q$–BIG CLIQUE is NP–complete for $q \geq \frac{1}{k}$, where $k > 2$. 

Marcin Mostowski, Jakub Szymanik
**Theorem**

For every rational number $0 < q < 1$ q–BIG CLIQUE is NP–complete

**Proof.**

Let $G = (V, E)$ be such that $\text{card}(V) = ka$. We show that in $G$ exists $\frac{1}{k}$–big clique iff in $G'$ exists $\frac{m}{k}$–big clique for $m < k$, where $G' = (V', E')$ is constructed as follows:

- $V' = V \cup U$, where $U$ such that $\text{card}(U) = n = \left\lceil \frac{(m-1)ka}{k-m} \right\rceil$ and $U \cap V = \emptyset$;
- $E' = E \cup U \times (U \cup V)$.

It suffices to observe that $\frac{n+a}{n+ka} \geq \frac{m}{k} > \frac{n+(a-1)}{n+ka}$. \qed
Ramsey quantifiers and their complexity

**Definition**

Let $f : \mathbb{N} \rightarrow \mathbb{N}$. $A$ is $f$–big set when $\text{card}(A) \geq f(\text{card}(U))$.

**Definition**

$M \models R_f x y \varphi(x, y)$ iff there is $f$–big $A \subseteq |M|$ such that for each $a, b \in A$, $M \models \varphi(a, b)$.

**Theorem**

Let $f_r(n)$ be the integer part of $rn$, for some rational $r$ such that $0 < r < 1$. Then $R_{f_r}$ is mighty.
**Future Work**

**Conjecture**

Let $k > 2$ and $0 < m < k$. There is PTIME class of graphs $J$ and NP-complete class $K \subseteq J$ s.t. for any $G \in J$ we have:

- $G \in K$ iff there is a clique in $G$ of size $\geq \frac{m}{k} \times \text{card}(G)$;
- $G \notin K$ iff there is no clique in $G$ of size $> \frac{m-1}{k} \times \text{card}(G)$.

**Conclusion**

Let $f$ be such that $\lim_{n \to \infty} \frac{f(n)}{n} = a$ exists and $0 < a < 1$. Then $R_f$ is NP-hard.

**Conclusion**

If $f$ satisfies the assumptions of the previous theorem and $f$ is PTIME computable, then $R_f$ is mighty.
Henkin quantifiers and complete problems, 
*APAL*, 32(1986).

[Mostowski and Wojtyniak, 2004] M. Mostowski and D. Wojtyniak
Computational complexity of the semantics of some natural language constructions, 

[Sevenster, 2006] M. Sevenster
Branching Imperfect Information. Logic, Language, and Computation, 