Characterizing Definability of Second-Order Generalized Quantifiers in Natural Language

Jakub Szymanik

Institute of Artificial Intelligence
University of Groningen

Göteborgs Universiteit
April 29, 2011

Abstract

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2. Some collective quantifiers are not definable in SO.
3. Then they can not be defined via the type-shifting strategy.
4. Everyday language doesn’t realize hard quantifiers.

Outline

Collectivity

(1.) All the Knights but King Arthur *met in secret*.

(2.) Most climbers *are friends*.

(3.) John and Mary *love each other*.

(4.) The samurai *were twelve in number*.

(5.) Many girls *gathered*.

(6.) Soldiers *surrounded* the Alamo.

(7.) Tikitu and Samson *lifted* the table.

Outline

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(2'.) $\exists X[X \subseteq \mathrm{Students} \wedge \mathrm{Play}(X)]$.

Existential modifier

Definition (Van Der Does 1992)

Fix a universe of discourse U and take any $X \subseteq U$ and $Y \subseteq \mathcal{P}(U)$. Define the existential lift Q^{EM} of a quantifier Q in the following way:

$$\mathrm{Q}^{EM}(X, Y) \text{ is true } \iff \exists Z \subseteq X[\mathrm{Q}(X, Z) \wedge Z \in Y].$$

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$$((et)((et)t)) \rightsquigarrow ((et)(((et)t)t))$$

Van Benthem problem

Observation

$(\cdot)^{EM}$ *works only for right monotone increasing quantifiers.*

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— $\downarrow$MON$\downarrow$ $\hookrightarrow$ $\uparrow$MON$\uparrow$

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(3.) $\exists A \subseteq$ Student[card$(A) = 5 \wedge$ Drink-a-whole-keg-of-beer(A)].

Neutral modifier

Definition (Van Der Does 1992)

Let U be a universe, $X \subseteq U$, $Y \subseteq \mathcal{P}(U)$, and Q a type (1, 1) quantifier. We define the *neutral modifier*:

$$\mathrm{Q}^N[X, Y] \text{ is true} \iff \mathrm{Q}\left[X, \bigcup(Y \cap \mathcal{P}(X))\right].$$

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$$\text{card}(\{x|\exists A \subseteq \text{Student}[x \in A \wedge \text{Drink-a-whole-keg-of-beer}(A)]\}) = 5$$

Determiner fitting

Definition (Winter 2001)

For all $X, Y \subseteq \mathcal{P}(U)$ we have that

$$\mathrm{Q}^{\mathrm{dfit}}(X, Y) \text{ is true}$$

$$\Longleftrightarrow$$

$$\mathrm{Q}[\cup X, \cup(X \cap Y)] \wedge [X \cap Y = \emptyset \vee \exists W \in X \cap Y \wedge \mathrm{Q}(\cup X, W)].$$

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$$\text{card}\big(\{x \in A | A \subseteq \text{Student} \wedge \text{Drink-a-whole-keg-of-beer}(A)\}\big) = 5$$
$$\wedge \exists W \subseteq \text{Student}[\text{Drink-a-whole-keg-of-beer}(W) \wedge \text{card}(W) = 5].$$

It really works . . .

Monotonicity of Q	Monotonicity of Q^{dfit}	Example
$\uparrow$MON$\uparrow$	$\uparrow$MON$\uparrow$	Some
$\downarrow$MON$\downarrow$	$\downarrow$MON$\downarrow$	Less than five
$\downarrow$MON$\uparrow$	$\sim$MON$\uparrow$	All
$\uparrow$MON$\downarrow$	$\sim$MON$\downarrow$	Not all
$\sim$MON$\sim$	$\sim$MON$\sim$	Exactly five
$\sim$MON$\downarrow$	$\sim$MON$\downarrow$	Not all and less than five
$\sim$MON$\uparrow$	$\sim$MON$\uparrow$	Most
$\downarrow$MON$\sim$	$\sim$MON$\sim$	All or less than five
$\uparrow$MON$\sim$	$\sim$MON$\sim$	Some but not all

Table: Monotonicity under the determiner fitting operator; cf. (Ben-Avi and Winter 2003).

. . . But violates invariance properties

Definition

A distributive determiner of type (1, 1) is conservative if and only if the following holds: for all M and all $A, B \subseteq M$:

$$\mathrm{Q}_M[A, B] \iff Q_M[A, A \cap B].$$

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Fact

For every Q *the quantifiers* Q^{EM}, Q^{N}, *and* Q^{dfit} *are not CONS.*

. . . And not only because of technicalities

Definition

We say that a collective determiner Q of type $((et)(((et)t)t))$ satisfies *collective conservativity* iff the following holds for all M and all $A, B \subseteq M$:

$$\mathrm{Q}_M[A, B] \iff \mathrm{Q}_M[A, \mathcal{P}(A) \cap B].$$

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Fact

For every Q the collective quantifiers Q^{EM}, Q^{N}, *and* Q^{dfit} *satisfy collective CONS.*

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Generalized quantifiers

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Characterizing definability of SOGQs

Collective majority

Discussion

Lindström quantifiers

Definition

Let $\tau = \{P_1, \ldots, P_r\}$ be a relational vocabulary, where P_i is l_i-ary for $1 \leq i \leq r$, and Q a class of τ-structures closed under isomorphisms. The class Q gives rise to a generalized quantifier which we also denote by Q. The tuple $s = (l_1, \ldots, l_r)$ is the *type* of the quantifier Q.

Examples Lindström quantifiers

$$
\begin{aligned}
\forall &= \{(A, P) \mid P = A\}. \\
\exists &= \{(A, P) \mid P \subseteq A \;\&\; P \neq \emptyset\}. \\
\text{even} &= \{(A, P) \mid P \subseteq A \;\&\; \text{card}(P) \text{ is even}\}. \\
\text{most} &= \{(A, P, S) \mid P, S \subseteq A \;\&\; \text{card}(P \cap S) > \text{card}(P - S)\}. \\
\text{M} &= \{(A, P) \mid P \subseteq A \text{ and } |P| > |A|/2\} \\
\text{some} &= \{(A, P, S) \mid P, S \subseteq A \;\&\; P \cap S \neq \emptyset\}.
\end{aligned}
$$

Logics with Lindström quantifiers

The extension FO(Q) is defined as usual.

$$\mathfrak{A} \models Q\overline{x}_1, \ldots, \overline{x}_r\, (\phi_1(\overline{x}_1), \ldots, \phi_r(\overline{x}_r)) \text{ iff } (A, \phi_1^{\mathfrak{A}}, \ldots, \phi_r^{\mathfrak{A}}) \in Q,$$

where $\phi_i^{\mathfrak{A}} = \{\overline{a} \in A^{l_i} \mid \mathfrak{A} \models \phi_i(\overline{a})\}$

Second-order structures

Definition

Let $t = (s_1, \ldots, s_w)$, where $s_i = (l_1^i, \ldots, l_{r_i}^i)$ is a tuple of positive integers for $1 \leq i \leq w$. A second-order structure of type t is a structure of the form $(A, P_1, \ldots, P_w)$, where $P_i \subseteq \mathcal{P}(A^{l_1^i}) \times \cdots \times \mathcal{P}(A^{l_{r_i}^i})$.

Second-order generalized quantifiers

Definition

A second-order generalized quantifier $\mathcal{Q}$ of type t is a class of structures of type t such that $\mathcal{Q}$ is closed under isomorphisms.

Examples of second-order GQs

$$
\begin{aligned}
\exists^2_1 &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \ \& \ P \neq \emptyset\}. \\
\textsc{Even} &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \ \& \ \mathrm{card}(P) \text{ is even}\}. \\
\textsc{Even}' &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \ \& \ \forall X \in P(\mathrm{card}(X) \text{ is even})\}. \\
\textsc{Most} &= \{(A, P, S) \mid P, S \subseteq \mathcal{P}(A) \ \& \ \mathrm{card}(P \cap S) > \mathrm{card}(P - S)\}. \\
\textsc{Most}^1 &= \{(A, P) \mid P \subseteq \mathcal{P}(A) \ \& \ \mathrm{card}(P) > 2^{\mathrm{card}(A)-1}\}.
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SOGQs do not decide invariance properties.

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Question

How invariance properties interact with definability?

FO($\mathcal{Q}$)

$$\mathfrak{A} \models \mathcal{Q}\overline{X}_1, \ldots, \overline{X}_w\,(\phi_1, \ldots, \phi_w) \text{ iff } (A, \phi_1^{\mathfrak{A}}, \ldots, \phi_w^{\mathfrak{A}}) \in \mathcal{Q},$$

where $\phi_i^{\mathfrak{A}} = \{\overline{R} \in \mathcal{P}(A^{l^i_1}) \times \cdots \times \mathcal{P}(A^{l^i_{r_i}}) \mid \mathfrak{A} \models \phi_i(\overline{R})\}$.

Warning

Do not confuse:

- FO GQs (Lindström) with FO-definable quantifiers

  E.g. most is FO GQs but is not FO-definable.
- SO GQs with SO-definable quantifiers

  E.g. MOST is SO GQs but not SO-definable.

GQs are not enough

Theorem (Kontinen 2002)

The extension $\mathcal{L}^*$ of first-order logic by all Lindström quantifiers cannot define the monadic second-order existential quantifier.

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Example

Some students gathered to play poker together.

GQs are not enough

Theorem (Kontinen 2002)

The extension $\mathcal{L}^*$ of first-order logic by all Lindström quantifiers cannot define the monadic second-order existential quantifier.

Example

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Corollary

Lindström quantifiers alone are not adequate for formalizing all natural language quantification.

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- Lifting first-order determiners
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For example . . .

Definition

Denote by some^{EM}:

$$\{(A, P, G) \mid P \subseteq A;\ G \subseteq \mathcal{P}(A) :\ \exists Y \subseteq P(Y \neq \emptyset \,\&\, P \in G)\}.$$

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(3.) Some students played poker together.

(3'.) some^{EM} $x, X[\text{Student}(x), \text{Play}(X)]$.

Another example . . .

Definition

We take five^{EM} to be the second-order quantifier denoting:

$$\{(A, P, G) \mid P \subseteq A;\ G \subseteq \mathcal{P}(A) :\ \exists Y \subseteq P(\mathrm{card}(Y) = 5 \,\&\, P \in G)\}.$$

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(4.) Five people lifted the table.
(4'.) five^{EM}$x, X[\mathrm{Student}(x), \mathrm{Lift}(X)]$.

SO-definable GQs are closed on lifts

Theorem

Let Q *be a Lindström quantifier definable in* SO*. Then the collective quantifiers* Q^{EM}, Q^{N}, *and* $\mathsf{Q}^{\mathrm{dfit}}$ *are definable in* SO*.*

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Proof.

Let us consider the case of Q^{EM}. Let $\psi(x)$ and $\phi(Y)$ be formulas. We want to express $\mathrm{Q}^{EM}x, Y(\psi(x), \phi(Y))$ in second-order logic. By the assumption, the quantifier Q can be defined by some sentence $\theta \in \mathrm{SO}[\{P_1, P_2\}]$. We can now use the following formula to simulate $\exists Z \subseteq X[\mathrm{Q}(X, Z) \wedge Z \in Y$:

$$\exists Z(\forall x(Z(x) \rightarrow \psi(x)) \wedge (\theta(P_1/\psi(x), P_2/Z) \wedge \phi(Y/Z)).$$

□

And this is the case for all SO-definable lifts

Theorem

Let us assume that the lift $(\cdot)^*$ *and a Lindström quantifier* Q *are both definable in second-order logic. Then the second-order generalized quantifier* Q^* *is also definable in second-order logic.*

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Corollary

Type-shifting strategy cannot take us outside SO.

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Definability—intuitions

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How do we formalize definability for SOGQs?

Example

$\exists^2_1$ is definable in $\mathcal{L}$ if there is a uniform way to express $\exists^2_1 X \psi(X)$ for any formula $\psi(X)$ in $\mathcal{L}$. Over a model $\mathfrak{A}$, $\psi(X)$ defines a collection of subsets $\{C \subseteq A \mid \mathfrak{A} \models \psi(C)\}$, so the problem is to find a way to express its non-emptyness for each $\psi(X)$.

$\mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$

Definition

Let $\mathcal{L}$ be a logic, $t = (s_1, \ldots, s_w)$ a second-order type, and let $\mathcal{G}_1, \ldots, \mathcal{G}_w$ be first-order quantifier symbols of types $s_1, \ldots, s_w$.

1. The models of $\mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ are of the form $\mathcal{A} = (\mathfrak{A}, G_1, \ldots, G_w)$, where $\mathfrak{A}$ is a first-order model and

$$G_i \subseteq \mathcal{P}(A^{l^i_1}) \times \cdots \times \mathcal{P}(A^{l^i_{r_i}}).$$

2. The quantifiers $\mathcal{G}_i$ are interpreted using the relations G_i:

$$\mathcal{A} \models \mathcal{G}_i \bar{x}_1, \ldots, \bar{x}_{r_i}(\phi_1(\bar{x}_1), \ldots, \phi_{r_i}(\bar{x}_{r_i}))$$

iff $(\phi_1^{\mathcal{A}}, \ldots, \phi_{r_i}^{\mathcal{A}}) \in G_i$.

Definability—definition

Observation

If $\phi \in \mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ *is a sentence of vocabulary* $\tau = \emptyset$*. Then*

$$\mathrm{Mod}(\phi) = \{(A, G_1, \ldots, G_w) \mid (A, G_1, \ldots, G_w) \models \phi\}$$

corresponds to a second-order generalized quantifier of type t*.*

Definition

Let $\mathcal{Q}$ be a quantifier of type t. The quantifier $\mathcal{Q}$ is definable in a logic $\mathcal{L}$ if there is $\phi \in \mathcal{L}(\mathcal{G}_1, \ldots, \mathcal{G}_w)$ of vocabulary $\sigma = \emptyset$ such that for any t-structure $(A, G_1, \ldots, G_w)$,

$$(A, G_1, \ldots, G_w) \models \phi \Leftrightarrow (A, G_1, \ldots, G_w) \in \mathcal{Q}.$$

Definability—basic facts

Theorem (Kontinen 2010)

If $\mathcal{Q}$ *is definable in* $\mathcal{L}$ *then* $\mathcal{L} \equiv \mathcal{L}(\mathcal{Q})$.

Definability—basic facts

Theorem (Kontinen 2010)

If $\mathcal{Q}$ is definable in $\mathcal{L}$ then $\mathcal{L} \equiv \mathcal{L}(\mathcal{Q})$.

Theorem (Kontinen 2010)

There is a quantifier $\mathcal{Q}$ of type $((1))$ *which is not definable in* FO *and satisfies* FO $\equiv$ FO($\mathcal{Q}$).

Characterizing definability—main idea

Recall, $\mathcal{Q}$ of type ((1)) is definable in SO if there is a sentence $\phi \in \mathrm{SO}(\mathcal{G})$ such that for all second-order structures (A, G):

$$(A, G) \models \phi \Leftrightarrow (A, G) \in \mathcal{Q}\,.$$

We show that SO and the relation G can be replaced by FO and a unary relation P by passing from A to a domain of cardinality $2^{|A|}$.

First-order encoding of second-order structures

Observation

1. *There is a one-to-one correspondence between integers $m \in B = \{0, \ldots, 2^n - 1\}$ and subsets of $A = \{0, \ldots, n - 1\}$;*
2. *Relations of A can be encoded as tuples of elements of B;*
3. *Sets of relations of A by relations of B.*

Formally

Definition

Let $t = (s_1, \ldots, s_w)$ be a type where $s_i = (1, \ldots, 1)$ is of length r_i for $1 \leq i \leq w$. Let $\mathfrak{A} = (A, G_1, \ldots, G_w)$ be a t-structure where $A = \{0, \ldots, n-1\}$ and $G_i \subseteq \mathcal{P}(A) \times \cdots \times \mathcal{P}(A)$. Denote by $\hat{\mathfrak{A}} = (B, P_1, \ldots, P_w)$ the following first-order structure of vocabulary $\tau = \{P_1, \ldots, P_w\}$, where P_i is a r_i-ary predicate, and

1. $B = \{0, \ldots, 2^n - 1\}$,
2. $P_i = \{(j_1, \ldots, j_{r_i}) \in B^{r_i} \mid (J_1, \ldots, J_{r_i}) \in G_i\}$, where, for $1 \leq k \leq r_i$, $\mathrm{bin}(j_k)$ is given by $s_0 \cdots s_{n-1}$, and $s_l = 1 \Leftrightarrow l \in J_k$.

$\mathcal{Q}^\star$

Definition

For a quantifier $\mathcal{Q}$ of type t, we denote by $\mathcal{Q}^\star$ the first-order quantifier of vocabulary τ defined by

$$\mathcal{Q}^\star := \{\hat{\mathfrak{A}} : \mathfrak{A} \in \mathcal{Q}\},$$

where $\hat{\mathfrak{A}}$ is the first-order encoding of $\mathfrak{A}$.

Characterization

Theorem

Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be monadic quantifiers. Then $\mathcal{Q}_1$ is definable in $\mathrm{MSO}(\mathcal{Q}_2, +)$ *if and only if $\mathcal{Q}_1^\star$ is definable in* $\mathsf{FO}(\mathcal{Q}_2^\star, +, \times)$.

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Built-in addition unleashes the expressive power of MSO.

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Collective MOST

(5.) Most groups of students played Hold'em together.

(5'.) $\textsc{Most}\, X, Y[\text{Students}(X), \text{Play}(Y)]$.

Question

Can we capture it via type-shifting?

Old result

Theorem

If the quantifier MOST *is definable in second-order logic, then counting hierarchy,* CH *is equal polynomial hierarchy,* PH*. Moreover,* CH *collapses to its second level.*

Old result

Theorem

If the quantifier MOST *is definable in second-order logic, then counting hierarchy,* CH *is equal polynomial hierarchy,* PH*. Moreover,* CH *collapses to its second level.*

Proof.

The logic FO(MOST) can define complete problems for each level of the CH (Kontinen&Niemisto'06). If MOST would be definable in SO, then FO(MOST) $\leq$ SO and therefore SO would contain complete problems for each level of the CH. This would imply that CH $=$ PH and furthermore that CH $\subseteq$ PH $\subseteq$ C_2P. ☐

MOST is not definable in SO

Theorem

The quantifier MOST^1 *is not definable in* SO*.*

MOST is not definable in SO

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Proof.

Show that definability of MOST^1 in SO implies that, for some k, the quantifier M is definable in FO$(+, \times)$ over cardinalities 2^{n^k}. Over these cardinalities, we could then express PARITY in the logic FO$(+, \times)$. This contradicts the result of Ajtai(1983). ☐

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Corollary

The type-shifting strategy is not general enough to cover all collective quantification in natural language.

Outline

- Introduction
- Lifting first-order determiners
- Generalized quantifiers
- Defining collective determiners
- Characterizing definability of SOGQs
- Collective majority
- Discussion

What is the right ontology for semantics?

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- $\mathcal{L}^*$ and SO don't capture natural language?
- Are many-sorted (algebraic) models more plausible?

Σ^1_1(Ristad's)-thesis

Σ^1_1-thesis

Our everyday language is semantically bounded by the properties expressible in the existential fragment of second-order logic.

- Does MOST belong to everyday language?
    - Everyday language doesn't realize prop. col. qua.
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Question

Did we just encounter an example where complexity restricts the expressibility of everyday language?

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- We can approach collectivity in terms of SOGQs.
- The previous attempts have relied on SO-definable GQs.
- They are not general enough.
- Complexity considerations suggest algebraic approach.

More details in:

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