Defining the meanings of quantifiers

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$\forall \quad \exists$
$\forall \quad \exists$

We search for possible mechanisms of understanding quantifiers in natural language.

Meaning of a natural language construction can be identified with a procedure of recognizing its extension.

Learning the semantics of natural language quantifiers consists essentially in collecting procedures for computing their denotations.

Given a natural language sentence we can recognize its truth–value using various semantic devices.

What is the nature of these semantic devices?

Referential meaning of a sentence φ is given by determining a method of establishing truth–value of φ in given possible situations.

Examples:

(1) Everyone in this room has read "The Chronicles of Narnia".

Referential meaning of 1: Ask everybody in this room whether she or he has read "The Chronicles of Narnia". If somebody says "NO" then 1 is false. Otherwise 1 is true.

(2) At least two people here speak polish.

Referential meaning of 2: Ask people one by one: "Czy znasz polski?". If two of them say "TAK", then 2 is true. If you ask everybody and do not find two answears "TAK", then 2 is false.

Some sentences are too hard for having practically plausible referential meanings in this sense.

The degree of understanding concrete constructions can be classified according to the corresponding sets of semantic procedures.

Natural language sentences can be ordered according to the degree of difficulty of deciding their truth values. Computational complexity could be one criterion.

First-order sentences are relatively easy, the problem of recognizing their truth value is in low complexity class.

An example of the **hard sentence** is the Hintikka's sentence:

(3) Some relative of each villager and some relative of each townsman hate each other.

Hintikka claimed that a logical form of 3 can not be expressed in first–order language. We need the Henkin quantifier to do this:

$$(4)\quad \begin{matrix}\forall x \exists y\\ \forall z \exists w\end{matrix}\,((V(x) \wedge T(z)) \Rightarrow$$

$$\Rightarrow (R(x,y) \wedge R(z,w) \wedge H(y,w))).$$

Theorem 1 (M. Mostowski, Wojtyniak 2004) *The problem of recognizing the truth–value of the Hintikka's sentence in finite models is* $NPTIME$*–complete.*

From Hintikka's sentence 3 we can infer that:

(5) Each villager has a relative
(6) $\forall x(V(x) \Rightarrow \exists y R(x,y))$.

This sentence can be false in an interpretation with an empty town. However, the Hintikka's formula 4 is true in every such interpretation. Therefore, the adequate logical form of Hintikka's sentence is given by the following formula with restricted branched quantifer:

$$(7) \qquad \begin{array}{l}(\forall x : V(x))(\exists y : R(x,y)) \\ (\forall z : T(z))(\exists w : R(z,w))\end{array} H(y,w)$$

which can be expressed in second–order logic as:

$$\exists S_1, S_2(\forall x(V(x) \Rightarrow \exists y(S_1(x,y) \wedge R(x,y)))\wedge$$

$$\wedge \forall z(T(z) \Rightarrow \exists w(S_2(z,w) \wedge R(z,w)))\wedge$$

$$\wedge \forall x,y,z,w((S_1(x,y) \wedge S_2(z,w)) \Rightarrow H(y,w))).$$

This improved reading has no influence on computational complexity of semantics of the Hintikka's sentence.

From:

Church's Thesis — the psychological version The computational mechanisms of mind do not differ essentially (are mutually reducible to each other in polynomial time) from the mechanisms of computation of Turing Machines.

Edmond's Thesis The class of practically computable problems is the same as the $PTIME$ class.

$P \neq NP$

follows that the mind is not equipped with any mechanism of recognizing NP–complete problems. But the problem of recognizing the truth–value of the Hintikka's sentence in finite models is NP–complete.

Easy sentences — sentences with practically plausible referential meanings.

Hard sentences — sentences without practically plausible referential meanings, e. g. Hintikka's sentence.

Having a hard sentence φ we can establish its truth–value by means of inferences (recognized by our logical competence) between φ and easy sentences. For example, knowing that an easy sentence ψ is true and $\psi \Rightarrow \varphi$ we know that φ is true; knowing that φ is false and $\psi \Rightarrow \varphi$ we know that ψ is false. In this way we determine **inferential meaning** of φ.

Examples of inferential meaning:

(8) At the party every girl was paired with a boy.
(9) Peter came alone to the party.
(10) Therefore: There were more boys than girls at the party.

Inferential meaning of the Hintikka's sentence:

(11) Each villager has an oldest relative.
(12) Each townsman has an oldest relative.
(13) The oldest relatives of each villager and of each townsman hate each other
(14) Therefore: Some relative of each villager and some relative of each townsman hate each other.

Definition 1 *A generalized (Lindström) quantifier* Q *of type* $(n_1, \ldots, n_k)$ *is a functor assigning to every set* X *a* k*-ary relation* $Q(X)$ *between relations on* X *such that if* $(R_1, \ldots, R_k) \in Q(X)$ *then* R_i *is an* n_i*-ary relation on* X*, for* $i = 1, \ldots, k$*. Additionally* Q *is preserved by bijection, i. e. if* $f : X \longrightarrow Y$ *is a bijection then* $(R_1, \ldots, R_k) \in Q(X)$ *if and only if* $(fR_1, \ldots, fR_k) \in Q(Y)$*, for every relations* $R_1, \ldots, R_k$ *on* X*, where* $fR = \{(f_1(x_1), \ldots, f_i(x_i)) : (x_1, \ldots, x_i) \in R\}$*, for* $R \subseteq X^i$*.*

Definition 2 *We say that a generalized quantifier* Q *is definable by second–order means if and only if there is a second–order formula* $\varphi(P_1, \ldots, P_n)$ *with only free variables* $P_1, \ldots, P_n$ *such that* $Q\bar{x_1} \ldots \bar{x_n}(\varphi_1(\bar{x_1}), \ldots, \varphi_n(\bar{x_n}))$ *is semantically equivalent to* $\varphi(\varphi_1(\bar{x_1}), \ldots, \varphi_n(\bar{x_n}))$*, for any* n*-tuple* $\varphi_1, \ldots, \varphi_n$ *of formulae such that no variable from* $\bar{x_i}$ *is bound in* φ_i*, for* $i = 1, \ldots, n$*, where* $\bar{x_i}$ *is a sequence of variables of the same length as the arity of* P_i *and all these sequences are mutually disjoint.*

The hierarchy of second–order formulae:

$\Sigma^1_0 = \Pi^1_0$ — only first–order quantifiers.

$\Sigma^1_{n+1} = \{\varphi : \text{there is } \psi \in \Pi^1_n \text{ s.t. } \varphi := \exists P_1 \ldots \exists P_k \psi\}$.

$\Pi^1_{n+1} = \{\varphi : \text{there is } \psi \in \Sigma^1_n \text{ s.t } \varphi := \forall P_1 \ldots \forall P_k \psi\}$.

At most three

$$\exists^{\leq 3} x \varphi(x)$$

$$\exists y_1 \exists y_2 \exists y_3 \forall x (\varphi(x) \Rightarrow (x = y_1 \vee x = y_2 \vee x = y_3)).$$

Even

$$\mathsf{D}_2 x \varphi(x)$$

$$\exists A \exists P [\forall x \forall y (P(x,y) \Rightarrow (A(x) \wedge \neg A(y)))$$

$$\wedge \forall x (A(x) \Rightarrow \exists y P(x,y)) \wedge$$

$$\wedge \forall y (\neg A(y) \Rightarrow \exists x P(x,y)) \wedge$$

$$\wedge \forall x \forall y \forall y' ((P(x,y) \wedge P(x,y')) \Rightarrow y = y' \wedge$$

$$\wedge \forall x \forall x' \forall y ((P(x,y) \wedge P(x',y)) \Rightarrow x = x')]$$

$$\exists R [\forall x R(x,x) \wedge \forall x \forall y (R(x,y) \Rightarrow R(y,x)) \wedge$$

$$\wedge \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \Rightarrow R(x,z)) \wedge$$

$$\wedge \forall x \forall y \forall z (R(x,y) \wedge R(x,z) \Rightarrow x = y \vee x = z \vee z = y) \wedge$$

$$\wedge \forall x \exists y (x \neq y \wedge R(x,y))]$$

Most

$$\text{MOST } x(\varphi(x), \psi(x))$$

$$\exists R[\forall x \exists y(\varphi(x) \wedge \psi(x) \wedge \varphi(y) \wedge \neg\psi(y) \wedge R(x,y)) \wedge$$

$$\wedge \forall x \forall y \forall y'(\varphi(x) \wedge \psi(x) \wedge \varphi(y) \wedge \neg\psi(y) \wedge \varphi(y') \wedge \neg\psi(y') \wedge$$

$$\wedge R(x,y) \wedge R(x,y') \Rightarrow y = y') \wedge$$

$$\wedge \neg\forall y \exists x(\varphi(y) \wedge \neg\psi(y) \wedge \varphi(x) \wedge \psi(x) \wedge R(x,y)) \wedge$$

$$\wedge \forall x \forall x' \forall y(\varphi(x) \wedge \psi(x) \wedge \varphi(x') \wedge \psi(x') \wedge \varphi(y) \wedge \neg\psi(y) \wedge$$

$$\wedge R(x,y) \wedge R(x',y) \Rightarrow x = x')].$$

Hintikka's form

$$\text{Z } xy(\varphi(x,y), \psi(x,y))$$

$$\exists A \exists B(\forall x \exists y(A(y) \wedge \varphi(x,y)) \wedge$$

$$\wedge \forall x \exists y(B(y) \wedge \varphi(x,y)) \wedge$$

$$\wedge \forall x \forall y(A(x) \wedge B(y) \Rightarrow \psi(x,y))].$$

There exist countably many

$$\exists^{=\aleph_0}$$

$$\exists R[\forall x \neg R(x,x) \wedge \forall x \forall y (R(x,y) \vee R(y,x) \vee x = y) \wedge$$

$$\wedge \forall x \forall y \forall z (R(x,y) \wedge R(y,z) \Rightarrow R(x,z)) \wedge$$

$$\wedge \forall A (\exists x A(x) \Rightarrow \exists x (A(x) \wedge \forall y R(y,x) \Rightarrow \neg A(y))) \wedge$$

$$\wedge \forall x (\exists y R(y,x) \Rightarrow \exists z (R(z,x) \wedge$$

$$\wedge \forall w (w \neq z \wedge R(w,x) \Rightarrow R(w,z))))].$$

Interpretation on arbitrary universes Weak semantics for second-order quantifiers

We consider structures of the form (M, K), where M is a model and K is a class of relations over $|M|$ closed on definability in a given language.

Proof system for Σ^1_1–quantifiers with assigned defining formulae:

$(L\psi 1)$

$$\frac{\psi(\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_k)}{Q\bar{x}(\psi_1, \dots, \psi_k)}$$

$(L\psi 2)$

$$\frac{Q\bar{x}(\psi_1, \dots, \psi_k)}{\psi(P_1, \dots, P_n, \psi_1, \dots, \psi_k)},$$

Definition 3 *We say that a sequence of second–order definable quantifiers* $Q_{\varphi_1}, \ldots, Q_{\varphi_m}$ *is a defining sequence for* Q *if and only if* Q *is* Q_{φ_m}*, that is* φ_m *is a defining formula for* Q *and for every* $i = 1, \ldots, m$ φ_i *is a* Σ^1_1*–formula in the language with additional quantifiers* $Q_{\varphi_1}, \ldots, Q_{\varphi_{i-1}}$*. Therefore,* φ_1 *must be a simple* Σ^1_1*-formula.*

Definition 4 *We say that a set* X *of second–order definable quantifiers is closed if and only if every quantifier in* X *has a defining sequence in* X*.*

Proposition 1 *For every second–order definable quantifier with a fixed defining formula there is a defining sequence.*

Definition 5 *Now we define the class of weak* $FO(X)$ *structures, where* X *is a class of second–order definable quantifiers, as the class of structures of the form* (M, K) *such that* K *is closed under definability by* $FO(X)$*–formulae.*

Theorem 2 (M. Mostowski 1995) *For every closed set* X *of second–order definable quantifiers exactly these formulae in the logic* $FO(X)$ *are* $FO(X)$*–provable which are* $FO(X)$*–tautologies.*

There is a common term ”natural language quantifiers” which is misleading. In a sense all known concrete quantifiers are natural language quantifiers.

It seems that the intension of using this term is the following: we would like to separate quantifiers expressible by simple constructions in everyday language.

What is the everyday language?

The notion is ambigous, but this is a language in which logicians communicate with bakers, students with postmen, quantum physicists with philosophers, etc.

Borderlines of the everyday language are changing. Its semantics is developing under strong influence of science, philosophy, theology, etc.

We learn about possible inferences, synonymy and others from our theoretical considerations. And then apply this knowledge in our communication.

Everyday language is used in our low–level communication. **Its constructions are strongly related by various procedures with real world.**

Σ^1_1–hypothesis: Everyday sentences are expressible in an existential fragment of second–order logic.

What is so special about Σ^1_1–sentences?

By Fagin's theorem Σ^1_1 properties of finite models are exactly NP properties of finite models. Therefore, to establish the truth–value of such sentences we can use non–deterministic algorithms.

For example, Hintikka's sentence is Σ^1_1–expressible as its improved logical form is:

$$\exists S_1, S_2(\forall x(V(x) \Rightarrow \exists y(S_1(x,y) \wedge R(x,y)))\wedge$$

$$\wedge\forall z(T(z) \Rightarrow \exists w(S_2(z,w) \wedge R(z,w)))\wedge$$

$$\wedge\forall x,y,z,w((S_1(x,y) \wedge S_2(z,w)) \Rightarrow H(y,w))).$$

Non–deterministic meaning: guess a witnesses relations S_1 and S_2 then check whether they satisfy the condition expressed by the first–order part of the above formula.

Argument for our hypothesis:

Barwise's test of the negation normality as a reasonable test for the first–order definability.

It was observed by J. Barwise that the negations of some simple quantifier sentences, i. e. sentences without sentential connectives different than "not" before a verb, can easily be formulated as simple quantifier sentences. For example:

(15) Everyone owns the car.
(16) Someone doesn't own the car.

In some cases it is impossible. Namely, the only way to negate some simple sentences is by prefixing them with the phrase ”it is not the case that” which has a metalinguistics nature. For example:

(17) Most relatives of each villager and most relatives of each townsman hate each other.

The sentences of the first kind are called negation normal. The first–order sentences are negation normal.

The test is based on the following fact:

Proposition 2 *If* φ *is a sentence definable in a* Σ^1_1 *existential fragment of second–order logic and its negation is logically equivalent to a* Σ^1_1*–sentence, then* φ *is logically equivalent to some first–order sentence.*

In other words, it works only on the assumption that simple everyday sentences are Σ^1_1–expressible.

What about going beyond everyday language?

What is the semantics for arbitrary second–order quantifiers?

Most of the authors considering semantics of natural language are interested only in finite universes. This drastic restriction probably follows from problems in treating in the intuitive way the semantics of some natural language constructions in infinite universes. Other reasons of restriction to finite models can be simplicity and algorithmisability. However, this is not enough to theoretically justify restricting ourselves to investigate only the case of finite universes. **Even if our world is finite we still can talk in our languages about infinite objects.** This is why we consider two cases: the semantics for second–order quantifiers in finite and arbitrary models.

Semantics for arbitrary second–order quantifiers in finite universes

Meaning of such sentences can be given in terms of **the alternating Turing Machine.**

Tree of computation with k alternations. Two players AND and OR. There is an accepting computation if the player OR has a winning strategy.

Definition 6 *The definition of the polynomial hierarchy runs inductively as follows:*

$$\Sigma_1^P = NP$$

$$\Sigma_{n+1}^P = NP^{\Sigma_n^P}$$

$$\Pi_n^P = co - \Sigma_n^P$$

$$PH = \bigcup_{i \geq 0} \Sigma_i^P$$

where $co - C$ *is the class of complements of problems in* C *(relative to appropriate alphabets).*

$PH \subseteq PSPACE$

Theorem 3 (Stockmeyer 1977) *For any* $n \in \omega - \{0\}$ Σ_n^1 *captures* Σ_n^P.

Semantics for arbitrary second–order quantifiers in arbitrary universes

In this case searching for witnesses is restricted by our cognitive abilities in more fundamental way.

This restriction can be described as restricting by definability (with parameters) in our language. Therefore, the relevant semantics for the second–order notions in the context of quantifiers in natural language can be the so–called **weak semantics**, proposed by M. Mostowski.

We consider structures of the form (M, K), where M is a model and K is a class of relations over $|M|$ closed on definability (with parameters) in a given language.

The class K is used to interpret second–order quantification. Phrases like: "for every relation R"', "there is a relation R" are interpreted in (M, K) as "for every relation R belonging to K" and "there is a relation R belonging to K".

This way of interpreting second–order quantification essentially modifies the standard semantics for second–order logic.

Our problem is to find the game for second–order quantifiers in arbitrary universes such that for a given model M:

Nature has a winnig strategy for φ if and only if for each K closed on definability in a given language $(M, K) \not\models \varphi$.

Me has winning strategy for φ if and only if for each K closed on definability in a given language $(M, K) \models \varphi$.