1 Fuzzy logic

A t-norm t is a binary function from $[0, 1]^2$ to $[0, 1]$ that is commutative, associative and monotone increasing with 1 as neutral element and 0 as zero element. That means that for arbitrary $x, y, z, u, v \in [0, 1]$ the following holds:

1. $xty = ytx$
2. $xt(ytz) = (xty)tz$
3. if $x \leq u$ and $y \leq v$, then $xty \leq utv$
4. $xt1 = x$ and $xt0 = 0$

Look at two arbitrary fuzzy sets $m_A$ and $m_B$ (where $m_A$ is a function assigning to every element of $D$ a number in $[0, 1]$). For any t-norm $t$ one can define the intersection $\cap_t$ for vague sets as follows:

$$m_{A \cap B}(x) = m_A(x)tm_B(x), \text{ for all } x \in D$$

(a) Show that $m_{A \cap B} \subseteq m_A$, where $m_A \subseteq m_B \iff \forall x \in D : m_A(x) \leq m_B(x)$.

(Hint: notice that from (3) and (4) it follows that $xty \leq xt1 = x$ and $xty \leq y$)

(b) Show that there is a t-function satisfying the above constraints such that $m_{A \cap B} \neq m_A$.

(c) Show that for all t-norms $t$ it holds that $xty \leq \min\{x, y\}$.

2 Contextual refinement

Let $R$ and $R^D$ be the relations ‘being visibly shorter than’ and ‘being indirectly visibly shorter than’, respectively, as defined in the handout given in the vagueness-class. Show the following:

1. $R \subseteq R^D$;
2. $R^D$ is irreflexive
3. $R^D$ is transitive, if $R$ is.
3 Semi-orders and equivalence relations

In class we showed how we can generate a linear order from a weak order $(I,>)$. First, we define the relation ‘~’ as $x \sim y$ iff $def \ x \not> y$ and $y \not> x$. Then we observed that ‘~’ is an equivalence relation that gives rise to the following set of equivalence classes: $\{[x]_\sim : x \in I\}$. Then we looked at the structure $\{(I,>)\}$, where ‘>’ was defined as follows: $X > Y$ iff $\exists x \in X : \exists y \in Y : x > y$. Then we proved that $\{([x]_\sim : x \in I), >^*\}$ is a linear order.

Let us now do something very similar for going from semi-orders $(I,>)$ to weak orders. We know that ‘~’ as defined as usual – $x \sim y$ iff $y \not> x$ – does not necessarily give rise to an equivalence relation. However, we can define the following relation: ‘≈’ as follows: $x \approx y$ iff $\exists x \in I : x \approx z$ iff $y \approx z$.

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(b) Give me an interval structure where (DIFF) does not hold.

4 Intervals and witnesses

Take an arbitrary interval structure $\Sigma_R = (I,<)$, where $I$ is a set of intervals and $<$ satisfies the conditions for interval orders. Define $x \subseteq y$ iff $\forall z[y < z \rightarrow x < z] \land \forall z[z < y \rightarrow z < x]$. It is easy to prove that ‘⊆’ is reflexive, transitive, and antisymmetric, and thus is a partial order. In terms of ‘<’ and ‘⊆’ we can now define three new principles, Convexity, Monotonicity, and Conjunction (the relation ‘∼’ is defined as usual):

(Conv) $x < y < z \rightarrow \forall u[x \subseteq u \land z \subseteq u \rightarrow y \subseteq u]$

(Mon) $x < y \rightarrow \forall z[z \subseteq x \rightarrow z < y]$

(Conj) $x \sim y \rightarrow \exists z \subseteq x[z \subseteq y \land \forall u \subseteq x[u \subseteq y \rightarrow u \subseteq z]$

(a) In class we showed that (Conv) holds in every interval order. What about (Mon) and (Conj)? If they hold, show me, if not, give a counterexample.

Now define the relations ‘begins before’, $<_B$, ‘ends before’, ‘$<_E$’, ‘begins at the same time’, ‘$=_B$’, and ‘ends at the same time’, ‘$=_E$’ as follows: (i) $x <_B y$ iff $\exists z[x < z \land z < y]$, (ii) $x <_E y$ iff $\exists z[x < z \land z \sim y]$, (iii) $x =_B y$ iff $x <_B y$ and $y <_B x$, and (iv) $x =_E y$ iff $x <_E y$ and $y \not<_E x$. If we now define the relation ‘∈’ in the expected way ($x \subset y$ iff $def \ x \subseteq y \land \forall z[y \not\subseteq x]$), we can formulate the following constraint:

(Diff) $(x \subset y \rightarrow x =_E y) \rightarrow \exists z[z \subset y \land z =_B y \land z \not<_E x]$

(b) Give me an interval structure where (Diff) does not hold.