Learnability of Quantifiers

Generalized Quantifiers

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Course Outline

(1) General introduction to generalized quantifiers
(2) Computational representations of quantifiers
(3) Machine learning for quantifiers 1
(4) Machine learning for quantifiers 2
(5) Cognitive perspective

Outline

Literature

- Westerståhl, Generalized Quantifiers, SEP.
- Peters & Westerståhl, Quantifiers in Language and Logic, OUP, 2008.
- Handbook of Logic and Language, Van Benthem & Ter Meulen (Eds.), Elsevier 2011.
- Szymanik, Quantifiers & Cognition, Studies in Linguistics & Philosophy, Springer, 2016.

Determiners: Examples

(1) **All** poets have low self-esteem.

(2) **Some** dean danced nude on the table.

(3) **At least 3** grad students prepared presentations.

(4) **An even number** of the students saw a ghost.

(5) **Most** of the students think they are smart.

(6) **Less than half** of the students received good marks.

(7) **Many** of the soldiers have not eaten for **several** days.

(8) **A few** of the conservatives complained about taxes.

And many more...

Determiners

Definition

Expressions that appear to be descriptions of quantity.

Example

All, not quite all, nearly all, an awful lot, a lot, a comfortable majority, most, many, more than *n*, less than *n*, quite a few, quite a lot, several, not a lot, not many, only a few, few, a few, hardly any, one, two, three.

Quantifiers are second-order relations

Observation

If we fix a model $\mathbb{M} = (M, A^M, B^M)$, then we can treat a generalized quantifier as a relation between relations over the universe.

Example

$$\text{every}[A, B] = 1 \text{ iff } A^M \subseteq B^M$$

$$\text{even}[A, B] = 1 \text{ iff } \text{card}(A^M \cap B^M) \text{ is even}$$

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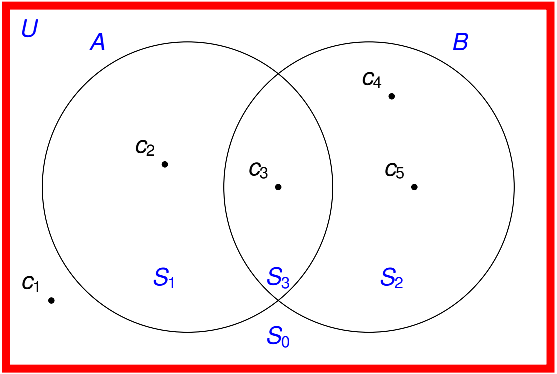
Illustration



Generalized Quantifiers

Definition

A quantifier Q is a way of associating with each set *M* a function from pairs of subsets of *M* into $\{0, 1\}$ (False, True).

Example

$$\text{every}_M[A, B] = 1 \text{ iff } A \subseteq B$$

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Space of GQs

Q is a function from M into a function from pairs of subsets of M into $\{0, 1\}$.

- If $\mathsf{card}(M) = n$, then there are $2^{2^{2n}}$ GQs.
- For $n = 2$ it gives 65,536 possibilities.

Question

Which of those are realized in natural language as determiners?

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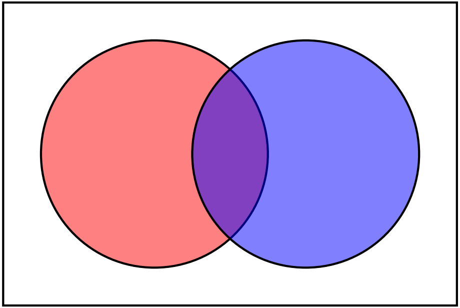
Isomorphism closure

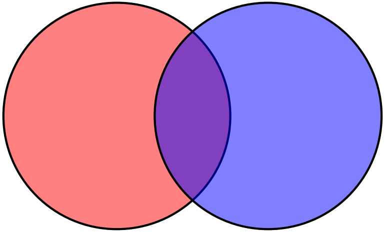
(ISOM) If $(M, A, B) \cong (M', A', B')$, then $Q_{\mathbf{M}}(A, B) \Leftrightarrow Q_{\mathbf{M'}}(A', B')$



Topic neutrality

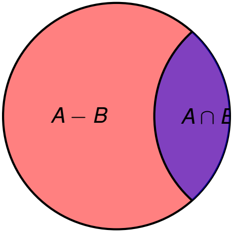
Extensionality

(EXT) If $M \subseteq M'$, then $Q_{\mathbf{M}}(A, B) \Leftrightarrow Q_{\mathbf{M'}}(A, B)$



Conservativity

(CONS) $Q_{\mathbf{M}}(A, B) \Leftrightarrow Q_{\mathbf{M}}(A, A \cap B)$



Monotonicity

↑MON $Q_M[A, B]$ and $A \subseteq A' \subseteq M$ then $Q_M[A', B]$.

↓MON $Q_M[A, B]$ and $A' \subseteq A \subseteq M$ then $Q_M[A', B]$.

MON↑ $Q_M[A, B]$ and $B \subseteq B' \subseteq M$ then $Q_M[A, B']$.

MON↓ $Q_M[A, B]$ and $B' \subseteq B \subseteq M$ then $Q_M[A, B']$.

Inference test

(1) Some boy is dirty.
(2) Some child is dirty.

(1) All children are dirty.
(2) All boys are dirty.

(1) All boys are muddy.
(2) All boys are dirty.

(1) No boy is dirty.
(2) No boy is muddy.

(1) Exactly five children are dirty.
(2) Exactly five boys are dirty.

The study of such ‘easy’ inferences on surface forms goes by the name ‘natural logic’ which is a thriving area of research

Monotonicity Universal

Monotonicity Universal (Barwise & Cooper 1981)

All simple determiners are monotone or conjunctions of monotone determiners

Research questions

- Do all NL determiners satisfy ISOM, EXT, MON, and CONS?
- Only? Every third? An even number of?
- Do all **simple** NL determiners satisfy ISOM, EXT and CONS?

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General definition

Definition

A monadic generalized quantifier of type (1,1) is a class Q of structures of the form $M = (U, A_1, A_2)$, where $A_1, A_2 \subseteq U$. Additionally, Q is closed under isomorphism.

Examples

$$\text{every} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \subseteq B\}.$$

$$\text{some} = \{(M, A, B) \mid A, B \subseteq M \text{ and } A \cap B \neq \emptyset\}.$$

$$\text{more than k} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \text{card}(A \cap B) > k\}.$$

$$\text{even} = \{(M, A, B) \mid A, B \subseteq M \text{ and } \text{card}(A \cap B) \text{ is even}\}.$$

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Boolean combinations

(1) At least 5 or at most 10 departments can win EU grants. (disjunction)

(2) Between 100 and 200 students started in the marathon. (conjunction)

(3) Not all students passed. (outer negation)

(4) All students did not pass. (inner negation)

Definition

$$(\mathsf{Q} \wedge \mathsf{Q}')_M[A, B] \iff \mathsf{Q}_M[A, B] \text{ and } \mathsf{Q}'_M[A, B] \text{ (conjunction)}$$
$$(\mathsf{Q} \vee \mathsf{Q}')_M[A, B] \iff \mathsf{Q}_M[A, B] \text{ or } \mathsf{Q}'_M[A, B] \text{ (disjunction).}$$
$$(\neg\mathsf{Q})_M[A, B] \iff \text{not } \mathsf{Q}_M[A, B] \text{ (complement)}$$
$$(\mathsf{Q}\neg)_M[A, B] \iff Q_M[A, M - B] \text{ (post-complement)}$$

Monotonicity interacts with negation

Theorem

Q *is MON*$\uparrow$

(1) iff $\neg$Q *is MON*$\downarrow$*.*

(2) iff Q$\neg$ *is MON*$\downarrow$*.*

Q *is* $\uparrow$*MON*

(1) iff $\neg$Q *is* $\downarrow$*MON.*

(2) iff Q$\neg$ *is* $\uparrow$*MON.*

Similarly for the downward monotone case.

Square of opposition

- some, $\neg$ some $=$ no, some $\neg$ $=$ not all, $\neg$ some $\neg$ $=$ all .
- some is ↑MON↑.
- Therefore, no is ↓MON↓, not all is ↑MON↓, and all is ↓MON↑.

Definability

Definition

Let Q be a generalized quantifier and $\mathcal{L}$ a logic. We say that the quantifier Q is *definable* in $\mathcal{L}$ if there is a sentence $\varphi \in \mathcal{L}$ such that for any $\mathbb{M}$:

$$\mathbb{M} \models \varphi \text{ iff } \mathsf{Q}_M[A, B].$$

Elementary GQs

Some GQs, like $\exists^{\leq 3}$, $\exists^{=3}$, and $\exists^{\geq 3}$, are expressible in FO (and therefore all boolean combinations thereof).

Example

$$\text{some } x\,[A(x), B(x)] \iff \exists x[A(x) \wedge B(x)]\,(\iff \mathbb{M} \in Q)$$

$$\text{less than two } x\,[A(x), B(x)] \iff \exists x \exists y\,[[A(x) \wedge B(x)] \wedge [A(y) \wedge B(y)] \rightarrow x = y]$$

Non-elementary GQs

Theorem

The quantifiers 'there exists (in)finitely many', most *and* even *are not first-order definable.*

We can use higher-order logics:

Example

In $\mathbb{M} = (M, A^M, B^M)$ the sentence

$$\text{most } x\,[A(x), B(x)]$$

is true if and only if the following condition holds:

$$\exists f : (A^M - B^M) \longrightarrow (A^M \cap B^M) \text{ such that f is injective but not surjective.}$$

Exercise. Write this definition of most as a formula in second-order logic with only the monadic predicate variables A and B free.

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How to Prove Undefinability?

Definition (Ehrenfeucht-Fraïsseé games for model-comparison)

Given two structures $\mathcal{A}$ and $\mathcal{B}$, and n.
We define a game between Spoiler and Duplicator:

1. Spoiler, picks either a member $a_1 \in \mathcal{A}$ or $b_1 \in \mathcal{B}$
2. Duplicator responds with picking a member from other structure
3. Spoiler and Duplicator continue to pick members for $n - 1$ more steps
4. Duplicator wins if the established mapping is a partial isomorphism

See also Six Lectures Ehrenfeucht-FraÃ¯ssÃ© games

How to Prove Undefinability

Lemma (Ehrenfeucht-Fraïssé)

Q is FO-definable iff $\exists k$ s.t.

$$\mathcal{M} \equiv_k \mathcal{M}' \Rightarrow \mathcal{M} \in Q \textit{ iff } \mathcal{M}' \in Q$$

where $\equiv_k$ means Dup has k-round winning strategy in E-F game.

In our case:

$$A \sim_k B \Leftrightarrow |A| = |B| = n < k \text{ or } |A|, |B| \geq k$$

and $\mathcal{M} \equiv_k \mathcal{M}'$ iff $A \cap B, A \setminus B, B \setminus A, M \setminus (B \cup A)$ all bear $\sim_k$ to their primed counterparts.

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Example: most

Theorem

most *is not FO-definable*

Proof.

Fix k. Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two models such that

- $|A_1 \cap B_1| = k + 1$, $|A_1 \setminus B_1| = k$
- $|A_2 \cap B_2| = k$, $|A_2 \setminus B_2| = k$

We have that $\mathbb{M}_1 \equiv_k \mathbb{M}_2$ but $\mathbb{M}_1 \in$ most and $\mathbb{M}_2 \notin$ most. □